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CHARACTERIZATIONS OF SOME CLASSES OF L^1-PREDUALS BY THE ALFSEN–EFFROS STRUCTURE TOPOLOGY

BY

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ABSTRACT

Using the Alfsen–Effros structure topology on the extreme boundary of the dual unit ball of a complex Banach space, we give characterizations of L^1-preduals (i.e., Banach spaces whose duals are isometrically isomorphic to $L^1(\mu)$ for a non-negative measure μ) and some of its subclasses viz. G-spaces, C_σ-spaces and $c_0(\Gamma)$ spaces.

Introduction

A closed subspace J of a complex Banach space X is called an L-ideal if there is a projection P from X onto J such that $\|x\| = \|Px\| + \|x - Px\|$ for all x in X. A closed subspace $M \subset X$ is called an M-ideal if $M^\perp = \{f \in X^* : f(M) = 0\}$ is an L-ideal in X^*. Let E denote the extreme points of the closed unit ball of X^*. A set $D \subset E$ is said to be structurally closed if there exists a w*-closed L-ideal N such that $N \cap E = D$. The sets D form the closed sets of a topology on E called the structure topology. The structure topology was introduced by Alfsen and Effros in [2]. We make use of properties of the structure topology and related results of M-ideal theory from [2] without explicitly mentioning them.

In this paper we study L^1-preduals (i.e., Banach spaces X such that $X^* = L^1(\mu)$, for some non-negative measure μ) and some subclasses of L^1-preduals using the structure topology. Most of our results directly depend on the results obtained in [8].

In Section 1, we develop the complex analogues of some of the results in [5]. Details of proofs are given only in those instances when they differ significantly from those in [5]. In Section 2, we characterize real, L^1-preduals as those Banach

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spaces X for which $\{f \in E : |f(x)| = 1\}$ is a structurally closed set for each $x \in X$. We then formulate and prove the corresponding result for complex Banach spaces.

For the definitions of G-spaces, C_σ, C_Σ spaces, see [15]. We freely use the characterizations of these spaces obtained in that paper. In Section 3 we characterize complex G-spaces as those Banach spaces X for which $|x| : E \to \mathbf{R}$, defined by $|x|(f) = |f(x)|$, is structurally upper semi-continuous for each $x \in X$. We attempt to solve a problem of Uttersrud [18], of characterizing G-spaces as those Banach spaces X for which the intersection of M-ideals is an M-ideal and line $\{f\}$ is an L-ideal for all $f \in E$. We show that if $\{f \in E$: line $\{f\}$ is an L-ideal$\}$ is w*-sequentially dense in the w*-closure of E, then line $\{f\}$ is an L-ideal for all f in the w*-closure of E. This enables us to give simple and transparent proofs of results which are more general than those obtained by N. Roy [17] and A. Gleit in [10], in this context.

C_σ-spaces are characterized in Section 4 as those Banach spaces X for which the structure topology agrees with the relative w*-topology on T-invariant subsets of E.

We make free use of concepts of convexity theory from [1].

NOTATIONS. For a complex Banach space X, let X_1 denote the closed unit ball, $S = \{x \in X_1 : \|x\| = 1\}$ and $S^* = \{f \in X_1^* : \|f\| = 1\}$. Let Z denote the w*-closure of E. For any $A \subset X^*$, let $c(A)$ denote the w*-closed, convex hull of A. Let $\mathbf{C}$ denote the complex plane and T the unit circle in $\mathbf{C}$.

For a compact convex set K (always considered in a locally convex Hausdorff topological vector space) let $E(K)$ denote the extreme boundary of K. For a probability measure μ on K, let $\gamma(\mu)$ denote the resultant of μ. Let $A(K)(A_{\mathbf{C}}(K))$ denote the space of real-valued (complex-valued) affine continuous functions on K, equipped with the supremum norm.

For a compact Hausdorff space Y let $C_{\mathbf{C}}(Y)$ denote the space of complex-valued continuous functions on Y. For a probability measure μ on Y, let Supp μ denote the topological support of μ. For any $y \in Y$, let $\delta(y)$ denote the Dirac measure at y.

For $p \in X_1^*$, let N_p denote the smallest w*-closed L-ideal containing p. *All closures, unless otherwise mentioned, are taken in the w^*-topology.* Let $\xrightarrow{w^*}$, $\xrightarrow{s}$ denote convergence in the w* and structure topologies respectively. For all other unexplained notations and terminology, see [2], [5], [8] and [15].

We will be using several times the following result from general Banach space theory (see [4], V. 5.9).

Let X be a Banach space and $K \subset X^*$ be a w*-closed convex set. Then line K (i.e., the linear span of K) is norm closed iff it is w*-closed.

1. *T*-faces and *T*-dilated sets

DEFINITION. Let X be a complex Banach space. For any $p, q \in X$, write $p < q$ if $\|q\| = \|q - p\| + \|p\|$.

The operation $<$ was introduced in [2] and is a partial ordering on X.

A set $H \subset X$ is said to be hereditary if $p \in H$, $q < p$ implies $q \in H$.

DEFINITION. A w*-closed, T-invariant convex set $H \subset X_1^*$ is called a T-face if (1) $\forall p \in H$, $p \neq 0$, $p/\|p\| \in H$; (2) H is hereditary.

Let H_p denote the smallest w*-closed T-face containing p.

REMARK 1. If $p \in E$ then $H_p = \{\lambda p : |\lambda| \leqq 1\}$. For any $0 \neq p \in X_1^*$, $E(H_p) \subset E$ and $H_p = H_{\lambda p}$ for $0 < \lambda < 1$.

DEFINITION. We say that a w*-closed set $D \subset X_1^*$ is T-dilated if for all $p \in D$, $E(H_p) \subset D$.

For the rest of this section we assume that X is an L^1-predual and $K = X_1^*$.

DEFINITION. Following the notation of [8], for any $p \in K$, define $\omega_p = R(\hom \mu)$, where μ is a maximal measure with $\gamma(\mu) = p$. Then by [6], ω_p is well defined and $\gamma(\omega_p) = p$.

REMARK 2. It is clear from the results in section 5 of [5] that for $0 \neq p \in K$, $\omega_p = \|p\| \cdot \omega p/\|p\|$ and if $\|p\| = 1$ then $\omega_p = \mu$ and $\operatorname{Supp} \omega_p \subset H_p$. Also a w*-closed, T-invariant set H is a T-face iff $\forall p \in H$, $p \neq 0$, $\operatorname{Supp} \omega_p \subseteq H$.

LEMMA 1.1. *For any T-face H of K, $N =$ line H is a w*-closed L-ideal.*

PROOF. We first claim that $N_1 = H$ and N is w*-closed.

Let $0 \neq p \in N_1$, then $p = \sum_{i=1}^n r_i q_i$, $q_i \in H$, $r_i \neq 0$ $\forall i$. So $p = \lambda \sum_{i=1}^n \lambda_i t_i q_i$ where $\lambda = \sum_{i=1}^n |r_i|$, $\lambda_i = |r_i|/\lambda$ and $t_i = r_i/|r_i|$ $\forall i$. Since $q = \sum_{i=1}^n \lambda_i t_i q_i \in H$, $q/\|q\| \in H$ and hence $p = \lambda \|q\| q/\|q\|$ is in H. Therefore $H = N_1$ and hence by the Krein–Šmulian theorem, we get that N is w*-closed.

That N is an L-ideal follows by an argument similar to the one used in the proof of the latter half of proposition 2.1 in [8].

LEMMA 1.2. *$c(D)$ is a T-face for any w*-closed, T-invariant set $D \subset K$ with the property that $\forall p \in D$, $\operatorname{Supp} \omega_p \subset c(D)$.*

PROOF. Let $0 \neq p \in c(D)$ and let μ be a probability measure on D with $\gamma(\mu) = p$ and let ν be a maximal measure on K dominating μ in the Choquet ordering. Then by theorem 2.1 of [5], $\exists$ nets of measures $\{\mu_j\}$ and $\{\nu_j\}$ such that $\mu_j \xrightarrow{w^*} \mu$, $\nu_j \xrightarrow{w^*} \nu$, $\mu_j = \sum_{i=1}^{n_j} c_i^j \delta(p_i^j)$, $\nu_j = \sum_{i=1}^{n_j} c_i^j \lambda_i^j$ for all j, where $0 \leqq c_i^j \leqq 1$ $\forall j, i$, $\sum_{i=1}^{n_j} c_i^j = 1$ $\forall j$ and $p_i^j \in D$ $\forall i$ and j, λ_i^j is a maximal measure with $\gamma(\lambda_i^j) = p_i^j$ $\forall j$ and i.

By hypothesis, $\operatorname{Supp} \omega p_i^j \subset c(D)$, i.e., $\operatorname{Supp} R(\hom \lambda_i^j) \subset c(D)$ and hence $\hom \lambda_i^j$ has its support in $c(D)$, as $\hom(R(\hom \lambda_i^j)) = \hom(\hom \lambda_i^j) = \hom \lambda_i^j$. So $\hom \nu_j$ has its support in $c(D)$. Since 'hom' is a w*-continuous map, we get that $\hom \nu$ has its support in $c(D)$. Therefore, $\operatorname{Supp} \omega_p = \operatorname{Supp} R(\hom \nu) \subset c(D)$. Now the proof is complete in view of Remark 2.

COROLLARY 1.3. *For any dilated set D, $c(D)$ is a T-face and $D \cap E$ is a structurally closed set.*

2. L^1-preduals

PROPOSITION 2.1. *Let X be a complex Banach space and an L^1-predual. If $x \in S$, then $\{f \in E : |f(x)| = 1\}$ is a structurally closed set.*

PROOF. Let $D = \{f \in Z : |f(x)| = 1\}$.

Let $p \in D \cap S^*$, then $\omega_p = \mu$ and since $\{f \in X_1^* : f(x) = 1\}$ is a w*-closed face, it is easy to see that $\operatorname{Supp} \mu \subset c(D)$. Now it follows from the results of the previous section that $N = \operatorname{line} c(D)$ is a w*-closed L-ideal and $N \cap E = \{f \in E : |f(x)| = 1\}$.

REMARK. The above proposition was proved in [5], when X is a real L^1-predual. However, the proof given there contains an error.

THEOREM 2.2. *Let X be a real Banach space such that for all $x \in S$, $\{f \in E : |f(x)| = 1\}$ is a structurally closed set, then X is an L^1-predual.*

PROOF. We use arguments similar to the ones used in proving theorem 3.3 of [8].

Let $x_0 \in S$ and let $F = \{f \in X_1^* : f(x_0) = 1\}$. Since $\{f \in E : |f(x_0)| = 1\} = N \cap E$, for some w*-closed L-ideal N, it is easy to see that $N = \operatorname{line} F$ and $N_1 = c(F \cup -F)$.

We claim that line F is an L-space.

Let $J = {}^{\perp}(\operatorname{line} F) = \{x \in X : f(x) = 0\ \forall f \in F\}$ and define $\Phi : X \mid J \to A(F)$ by $\Phi(x + J)(f) = f(x)$, $f \in F$, $x \in X$. Since $\Phi(x_0 + J) = 1$ (the constant function) using a standard argument in convexity theory it is easy to see that Φ is an onto isometry.

Let $a \in A(F)$, $\|a\| = 1$ and let $G = \{f \in F : a(f) = 1\}$. Since Φ is onto $\exists x_1 \in X \ni \|x_1 + J\| = 1$ and $\Phi(x_1 + J) = a$. Since J is an M-ideal $\exists x_2 \in J \ni \|x_1 + x_2\| = \|x_1 + J\| = 1$. Now if $G = \{f \in X_1^* : \frac{1}{2}f(x_0 + x_1 + x_2) = 1\}$ then it is easy to see that line G is a w*-closed L-ideal in line F and consequently G is a split face of F. From [7], it then follows that F is a simplex and hence line F is an L-space.

Let $f_0 \in E$ and let $0 < \varepsilon < 1$. Using the Bishop–Phelps theorem get $g_0 \in X_1^*$ such that $\|f_0 - g_0\| \leqq \varepsilon$ and $\|g_0\| = g_0(y)$, $y \in S$. By what we have seen above, if $G = \{f \in X_1^* : f(y) = 1\}$ then line G is a w*-closed L-ideal and $g_0 \in$ line G. If P denotes the L-projection from X^* onto line G, then since $\|(I - P)(f_0 - g_0)\| < \varepsilon$ and $P(f_0) = 0$ or f_0, we get that $P(f_0) = f_0$. Therefore $f_0 \in$ line G and hence is an extreme point of the unit ball of the dual L-space line G. Hence by theorem 5.8 of [13], line $\{f_0\}$ is an L-ideal in line G and consequently in X^*.

If $f_0 \in S^*$ and $P(f_0) = 0$ or f_0 for all L-projections P, then we get g_0 and G as above and observe that $g_0 \in$ line G and apply theorem 5.8 of [13] to conclude that $f_0 \in E$.

Applying theorem 5.8 of [13] once again we get that X is an L^1-predual.

REMARK. The hypotheses of the above theorem is equivalent to saying that line F_x is an L-ideal and $(\text{line}\, F_x)_1 = c(F_x \cup -F_x)$ $\forall x \in S$, where $F_x = \{f \in X_1^* : f(x) = 1\}$.

An example was mentioned in section 4 of [8] to the effect that the above result is not true when X is a complex Banach space. We give below a partial complex analogue. First we recall the definition of an M-set from [8].

DEFINITION. Let $X \subset C_{\mathbb{C}}(Y)$ be a closed subspace separating points of Y. A closed set $D \subset Y$, of the form $D = e^{-1}(TF)$, where F is a w*-closed face of X_1^* and $e : Y \to X_1^*$ is the evaluation map, is called an *M-set* if for any boundary measure μ on Y (i.e., $|\mu| \circ e^{-1}$ is a maximal measure on X_1^*) and $\mu \in A^{\perp}$, we have $\mu \mid D \in A^{\perp}$.

THEOREM 2.3. *Let $X \subset C_{\mathbb{C}}(Y)$ be a closed subspace, separating points of Y. The following are equivalent.*

(1) *For all $f_0 \in S$, $\{a \in E : |a(f_0)| = 1\}$ is structurally closed and if $F = \{a \in X_1^* : a(f_0) = 1\}$ then F is split in $c(F \cup -iF)$.*

(2) *For all $f_0 \in S$, $D = \{y \in Y : |f_0(y)| = 1\}$ is an M-set and if $B = \{\bar{f}_0 g : g \in X\}$ then $B \mid D$ is a closed self-adjoint subspace of $C_{\mathbb{C}}(D)$ ($\bar{f}_0$ stands for the complex conjugate of f_0).*

(3) *X is an L^1-predual.*

PROOF. We shall prove that (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2): We freely use the notations introduced in the proof of theorem 3.3 of [8].

Let $F = \{a \in X_1^* : a(f_0) = 1\}$. Since $D = e^{-1}(TF)$, it follows from the proof of (1) $\Rightarrow$ (2) of theorem 3.3 in [8] that D is an M-set.

It follows from the results in section 4 of [15] that the map $\Phi : X \mid J \to A_C(F)$, defined in the canonical way ($J = {}^{\perp}(\text{line } F)$), is an onto isometry. Let $f, g \in X$ be such that $\Phi(\overline{f+J}) = \Phi(g+J)$. For any $y \in D$, if $e(y) = t\mathcal{T}$, $\mathcal{T} \in F$ and $t \in T$ then we have $f_0(y) = t$ and $\overline{f_0(y)}g(y) = \Phi(g+J)(\mathcal{T}) = \Phi\overline{(f+J)}(\mathcal{T}) = f_0(y)\overline{f(y)}$. There $\bar{f}_0 g$ is the conjugate of $\bar{f}_0 f$ on D. Hence $B \mid D$ is self-adjoint.

It is trivial to verify that $B \mid D$ is isometrically isomorphic to $X \mid J$ and hence $B \mid D$ is closed.

(2) $\Rightarrow$ (1): If we define $Q : X^* \to X^*$ by $Q(p)(f) = \int_D f d\mu$ for $f \in X$, $p \in X^*$ and μ a boundary measure on Y representing p with $\|p\| = \|\mu\|$, then the same reasoning as in the proof of theorem 3.3 of [8] yields that line F is a w*-closed L-ideal and $(\text{line } F)_1 = c(TD)$. Therefore $\{a \in E : |a(f_0)| = 1\}$ is structurally closed. Using the self-adjointness of $B \mid D$, it is easy to see that $\Phi : X \mid J \to A_C(F)$ has self-adjoint range, where Φ, F and J are defined as before. Since the range of Φ contains the constant function 1, it must coincide with $A_C(F)$. Hence F is split in $c(F \cup -iF)$ (see [11], chapter 7, lemma 12).

(1) $\Rightarrow$ (3): Let $f_0 \in S$, $F = \{a \in X_1^* : a(f_0) = 1\}$. Using the hypotheses it is easy to see that line F is a w*-closed L-ideal and $(\text{line } F)_1 = c(TF)$. Hence the natural map $\Phi : X \mid J \to A_C(F)$ (where $J = {}^{\perp}(\text{line } F)$) is an isometry and the range of Φ contains the constant function 1. We claim that Φ is onto. It is enough to show that the range to Φ is a self-adjoint subspace of $A_C(F)$. But this again follows from lemma 12, chapter 7 of [11], since F is split in $c(F \cup -iF)$. Hence as in the proof of Theorem 2.1, we get that line F is a dual L-space. Now an argument similar to the one given in the latter half of the proof of Theorem 2.1 completes the proof.

3. *G*-spaces

THEOREM 3.1. *Let X be a complex Banach space. Then X is a G-space iff $\forall x \in X$, $|x| : E \to \mathbf{R}$, defined by $|x|(f) = |f(x)|$ $\forall f \in E$, is upper semi-continuous (u.s.c.) in the structure topology.*

PROOF. Suppose $|x|$ is structurally u.s.c. $\forall x \in X$. Let $0 \neq f_0 \in Z - E$ and let $\{f_i\}$ be any net in E and $f_i \xrightarrow{w^*} f_0$.

Fix $p_0 \in N_{f_0} \cap E$ and let $0 \neq x \in X$, $c > 0$ and $|p_0(x)| < c$. Then by hypothesis $\{f \in E : |f(x)| < c\}$ is a structurally open set containing p_0. Since by lemma 3.8 of [2, part II], $f_j \xrightarrow{s} p_0$, there is a j_0 such that $j \geqq j_0$ implies $|f_j(x)| < c$. As $f_j(x) \to f(x)$, it follows that $|f(x)| \leqq c$. Hence line$\{p_0\}$ = line$\{f_0\}$ and this is true for all $p_0 \in N_{f_0} \cap E$. Therefore N_{f_0} = line$\{f_0\}$ and hence line$\{f_0\}$ is an L-ideal.

Now let $D \subset E$ be any w*-compact, T-invariant set. Let $\{f_j\}$ be any net in D and $f_j \xrightarrow{s} f$, $f \in E$. By replacing $\{f_j\}$ by a subnet if necessary, we may assume that $f_j \xrightarrow{w^*} g_0$ and $g_0 \in D$. Using an argument similar to the one above we can see that $|g_0(x)| \leqq |f_0(x)|$ $\forall x \in X$. Since $\|f_0\| = 1 = \|g_0\|$, we get that $f_0 = \alpha g_0$, $\alpha \in T$. Since D is T-invariant, $f_0 \in D$. Hence D is structurally closed.

Since $\forall f \in Z$, line $\{f\}$ is an L-ideal, we have $Z \subset [0,1]E$. Hence $S^* \cap Z = E$. Since S^* is a G_δ in the relative w*-topology of X_1^*, we get that E is a Borel set. Also for any maximal measure μ on X_1^*, $\mu(S^*) = 1$ and $\mu(Z) = 1$ imply $\mu(E) = 1$. Hence X_1^* is a 'standard' compact convex set.

It now follows from the proof of theorem 2.2 in [8] that X is an L^1-predual. Therefore X is a G-space.

If X is a G-space, then it is not difficult to see that $|x|$ is u.s.c. in the structure topology (see [16]).

Corollary 3.2. *If X is a complex Banach space and the structure topology on E is such that for any p_1, $p_2 \in E$, p_1, p_2 linearly independent, there exist disjoint structurally open sets separating p_1 and p_2, then X is a G-space.*

Proof. Let $0 \neq x \in X$ and $c > 0$, let $D = \{f \in E : |f(x)| \geqq c\}$. If $\{f_j\}$ is any net in D and $f_j \xrightarrow{s} f$, $f \in E$, we may assume that $f_j \xrightarrow{w^*} g$, $g \in Z$. Since $|f_j(x)| \geqq c$ $\forall j$, we have $g \neq 0$. Hence by lemma 3.8 of [2, part II] and the separation property assumed in the hypothesis it follows that $f = tg/\|g\|$, $t \in T$. Hence

$$|f(x)| = \frac{1}{\|g\|}|g(x)| \geqq c.$$

Therefore $f \in D$.

Hence $|x|$ is u.s.c. in the structure topology.

Remark. Corollary 3.2 was proved for real Banach spaces in [18] by a different argument.

Corollary 3.3. *A compact convex set K is a Bauer simplex if and only if for all $a \in A(K)$, $|a| : E(K) \to \mathbf{R}$ is u.s.c. in the facial topology.*

Proposition 3.4. *Let X be a complex Banach space. Consider the following statements.*

(1) *X is a G-space.*

(2) $\forall x \in X$, $x \neq 0$ $U_x = \{f \in E : f(x) \neq 0\}$ *is open in the structure topology.*

(3) $\forall f \in E$, line$\{f\}$ *is an L-ideal and the intersection of any family of M-ideals in X is an M-ideal.*

(4) *For any $D \subset E$, $\overline{\text{line}\, D}$ is an L-ideal.*

Then we have $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.

PROOF. Follows easily from lemma 5 of [16] and remark 2 after the proof of theorem 10 of [18].

COROLLARY 3.5. *Let $X = \{f \in C_C(Y) : f(y_a) = \lambda_a t_a f(z_a);\ t_a \in T,\ \lambda_a \in [0,1]$ and $y_a, z_a \in Y$, $a \in \Sigma\}$ (where Σ is an index set). Then for any $D \subset Y$,*

$$M_D = \{f \in X : f(D) = 0\} \text{ is an } M\text{-ideal in } X.$$

PROOF. Let $e : Y \to X_1^*$ denote the evaluation map. Using theorem 24 of [15], we see that for all $y \in Y$, $e(y) \in [0,1]E$ and line $\{e(y)\}$ is an L-ideal. Now use the above proposition to conclude that $M_D = \bigcap_{y \in D} \{f \in X : f(y) = 0\}$ is an M-ideal in X.

REMARK. In the above proposition, whether $(3) \Rightarrow (1)$ or not seems to be still an open problem. In [17], Roy has proved that $(3) \Rightarrow (1)$ if X is a separable, real, L^1-predual. We next give a simple proof of the complex analogue of Roy's result in a somewhat general setting. Lima et al. [14] have also given a simple proof of Roy's result.

PROPOSITION 3.6. *Let X be a Banach space such that*

(1) $A = \{f \in E;$ line $\{f\}$ *is an L-ideal$\}$ is sequentially w^*-dense in Z,*

(2) *the intersection of any countable family of M-ideals in X is an M-ideal.*

Then for any $f \in Z$, line $\{f\}$ *is an L-ideal.*

PROOF. Let $f \in E - A$ and choose a sequence $\{f_n\}$ in A such that the f_n's are T-independent, i.e., $f_n \neq t \cdot f_m$ for any $t \in T$ and n, m and $f_n \xrightarrow{w^*} f$. (This can be done since A is a T-invariant set and $f \neq 0$, only for finitely many n's, $f_n \in T \cdot \{f_i\}_{i=1}^{\infty}$ and redefining the sequence by discarding those finite n's).

Let $N = \overline{\text{line}}\{f_n\}_{n=1}^{\infty}$ (closure in the norm topology). Since the f_n's are T-independent and line$\{f_n\}$ is an L-ideal for all n, we have

$$N = \left\{ \sum_{i=1}^{\infty} \alpha_i f_i : \sum_{i=1}^{\infty} |\alpha_i| < \infty \right\} \quad \text{and} \quad \left\| \sum_{i=1}^{\infty} \alpha_i f_i \right\| = \sum_{i=1}^{\infty} |\alpha_i|.$$

Let $F = c\{f_n\}_{n=1}^{\infty}$. Since $E(F) = \{f_n\}_{n=1}^{\infty} \cup \{f\}$, it is easy to see that line $F = N + \text{line}\{f\}$. Since N is norm closed, line F is norm closed and hence is w*-closed. Also line $F=(\bigcap_n \operatorname{Ker} f_n)^{\perp}$ (Ker stands for the Kernel). Therefore line F is a w*-closed L-ideal.

Since f is an extreme point T-independent of $\{f_n\}_{n=1}^{\infty}$ we get that $f \notin N$ and $\|\lambda f+\sum_{i=1}^{\infty}\lambda_i f_i\| = |\lambda| + \sum_{i=1}^{\infty}|\lambda_i|$. Therefore line$\{f\}$ is an L-ideal in line F and hence in X^*.

Now let $0 \neq f \in Z-E$.

Case (*i*): Assume that X_1^* is a standard compact convex set. If $f/\|f\| \notin E$, choose a maximal measure ν with $\gamma(\nu)=f/\|f\|$ and $p_1, p_2 \in \operatorname{Supp} \nu \cap E$, p_1, p_2 independent. We choose a sequence $\{f_n\}$ in $E-T\{p_1,p_2\}$ such that the f_n's are T-independent and $f_n \xrightarrow{w^*} f$.

As before let $N=\{\sum_{i=1}^{\infty}\lambda_i f_i : \sum_{i=1}^{\infty}|\lambda_i|<\infty\}$ and $F=c\{f_n\}_{n=1}^{\infty}$.

If $E(F)=\{f_n\}_{n=1}^{\infty}$ then since $f \in N$ we have $N=$ line F. Hence line F is a w*-closed L-ideal. On approximating ν by simple measures having resultant $f/\|f\|$, it is easy to see that $\operatorname{Supp} \nu \subset$ line F. Hence $p_1, p_2 \in N$, contradicting the choice of the sequence $\{f_n\}$.

If $E(F)=\{f_n\}_{n=1}^{\infty} \cup \{f\}$ then line $F=N+$ line$\{f\}$ and we may assume that $f \notin N$.

As before, line F is a w*-closed L-ideal and $p_1, p_2 \in$ line $F=N \oplus$ line$\{f\}$. Hence $p_2 \in N \oplus$ line$\{p_1\}$. Since p_1 is an extreme point T-independent of $\{f_n\}_{n=1}^{\infty}$, this direct sum is a L^1-direct sum. Therefore $p_2 \in N$ or $p_2 \in T\{p_1\}$, again contradicting the choice of the sequence $\{f_n\}$ and points p_1, p_2.

Therefore $f/\|f\| \in E$ and hence line$\{f\}$ is an L-ideal.

Case (*ii*): X is arbitrary.

Let $\{f_n\}$ be any T-independent sequence in E with $f_n \xrightarrow{w^*} f$. If $F=c\{f_n\}_{n=1}^{\infty}$, it is easy to see that $M=$ line F is a w*-closed L-ideal and a separable dual Banach space and hence M_1 is a standard compact convex set.

If $D \subset E(M_1)$ then $D \subset E$ and D has only countably many T-independent vectors. Using the hypotheses and Proposition 3.4, we see that the separable Banach space $X/{}^{\perp}M$ satisfies the same hypotheses as the X in case (i). Therefore $f/\|f\| \in E(M_1) \subset E$.

Hence line $\{f\}$ is an L-ideal.

COROLLARY 3.7. *If X is an L^1-predual space with the property that E is w^*-sequentially dense in Z and the intersection of any countable family of M-ideals is an M-ideal, then X is a G-space.*

COROLLARY 3.8. *If K is a compact convex set such that*

(i) $A=\{x \in E(K) : \{x\}$ *is a split face*$\}$ *is sequentially dense in* $\overline{E(K)}$,

(ii) *the intersection of any family of M-ideals in $A(K)$ is an M-ideal in $A(K)$, then K is a Bauer simplex.*

PROOF. We use the above proposition and the correspondence between closed split faces of K and w*-closed L-ideals of $A(K)^*$ to deduce that $E(K)$ is closed (noting that $1 \in A(K)$) and $\{x\}$ is a split face $\forall x \in E(K)$.

Let $a_0 \in A(K)^+$, $a_0 \neq 0$ and let $F = \{x \in K : a_0(x) = 0\}$. Let $M = \{a \in A(K) : a(x) = 0 \ \forall x \in E(F)\}$. Since $E(F) \subset E(K)$ and $\{x\}$ is a split face $\forall x \in E(K)$, we get by hypotheses that M is an M-ideal. Hence $M = \{a \in A(K) : a(G) = 0\}$ for some closed split face G of K.

$a_0 \in M \Rightarrow a_0(G) = 0 \Rightarrow G \subset F$

If $x \in E(F) - G$ then since line $e(G)$ is w*-closed (e is the evaluation map) and $e(x) \notin \text{line}\, e(G)$, $\exists a \in A(K) \ni a(G) = 0$ and $a(x) \neq 0$. This contradiction shows that $G = F$. Hence any peak face of K is a split face. From [7], it follows that K is a simplex. Hence K is a Bauer simplex.

REMARK. The set K considered in proposition II.3.17 of [1] provides an example of a non-metrizable compact convex set K for which $E(K)$ is sequentially dense in $\overline{E(K)}$. Corollary 3.8 improves a result of A. Gleit [10].

4. Other classes of L^1-preduals

THEOREM 4.1. *For a complex Banach space X, the following are equivalent.*

(1) *X is a C_σ-space.*

(2) (i) *$A = \{f \in E : \text{line}\{f\}$ is an L-ideal$\}$ is w^*-dense in Z,* (ii) *for any L-ideal $N \subset X^*$, $\bar{N}$ is an L-ideal and $(\bar{N})_1 = \overline{(N_1)}$.*

(3) *Any relatively w^*-closed, T-invariant subset of E is structurally closed.*

(4) *For all $x \in X$, $|x|$; $E \to \mathbf{R}$ is lower semi-continuous in the structure topology.*

PROOF. (1) $\Rightarrow$ (2): Since $\forall f \in E$, line $\{f\}$ is an L-ideal, (i) is clear. Let $N \subset X^*$ be any L-ideal and let $D = \bigcup_{p \in N \cap S^*} \text{Supp}\, \mu_p$, where μ_p is the unique maximal measure with $\gamma(\mu_p) = p$.

Let $p \in N \cap S^*$. On approximating μ_p by simple measures having resultant p, it is easy to see that $\text{Supp}\, \mu_p \subset \overline{(N_1)}$. Hence $\overline{(N_1)} = c(TD)$.

Since X is a C_σ-space, $T\bar{D}$ is a dilated set. Therefore by Corollary 1.3 and Lemma 1.1, we get that $\bar{N} = \text{line}\, c(TD)$ is a w*-closed L-ideal and $(\bar{N})_1 = \overline{(N_1)}$.

(2) $\Rightarrow$ (3): Let $f \in E$, $\{f_i\}$ be a net in A and $f_i \xrightarrow{w^*} f$. Let $D_j = \{f_i\}_{i \geq j}$ and $N_j = \overline{\text{line}\, D_j}$ (closure in the norm topology). Since line $\{f_i\}$ is an L-ideal $\forall i$, we get

that N_j is an L-ideal. Therefore $\bar{N}_j$ is an L-ideal and $(\bar{N}_j)_1 = \overline{((N_j)_1)}$. Let $p \in N_j \cap S^*$, $\{p_n\} \subset (\text{line } D_j)_1$ and $p_n \to p$ in norm. It is easy to see that p_n's are in the convex hull of TD_j. Therefore $p \in c(TD_j)$ and hence $(\bar{N}_j)_1 = c(TD_j) = \overline{((N_j)_1)}$.

Now let $N = \bigcap_j \bar{N}_j$. Then N is a w*-closed L-ideal and line$\{f\} \subset N$. If $g \in N \cap S^*$ then $g \in (\bar{N}_j)_1$ $\forall j$ so that $g \in \bigcap_j c(TD_j)$. Now if $g \notin$ line$\{f\}$, then $\exists 0 \neq x_0 \in X \ni g(x_0) \neq 0$ and $f(x_0) = 0$. For any $\varepsilon > 0$ $\exists j_\varepsilon \ni j \geqq j_\varepsilon \Rightarrow |f_j(x_0)| < \varepsilon$. Therefore for any $h \in c(TD_{j_\varepsilon})$, $|h(x_0)| \leqq \varepsilon$. In particular $|g(x_0)| \leqq \varepsilon$. As ε is arbitrary, we get a contradiction. Hence $N =$ line$\{f\}$ and consequently line$\{f\}$ is an L-ideal.

For any $D \subset E$, relatively w*-closed and T-invariant, let $N = \overline{\text{line } D}$ (closure in norm topology). By what we have seen above, N is an L-ideal. Hence by hypotheses we get that $\bar{N}$ is an L-ideal and $D = \bar{D} \cap E = c(D) \cap E = \overline{(N_1)} \cap E = (\bar{N})_1 \cap E$. Therefore D is a structurally closed set.

(3) $\Rightarrow$ (4) This is easy to see.

(4) $\Rightarrow$ (1): Let $0 \neq f \in Z - E$ and let $\{f_j\}$ be a net in E, $f_j \xrightarrow{w^*} f$. Fix $p \in N_f \cap E$ and let $0 \neq x \in X$, $c > 0$ and $|p(x)| > c$, Then $\{g \in E : |g(x)| > c\}$ is a structurally open set containing p. Since $f_j \xrightarrow{s} p$, we get that $|f(x)| \geqq c$. Now as in the proof of Theorem 3.1, we get that line$\{f\}$ is an L-ideal.

Using arguments similar to the ones used in the proof of Theorem 3.1, we can see that X is a G-space and as a consequence $|x|$ is u.s.c. in the structure topology. Therefore $|x|$ is continuous in the structure topology for all x.

Finally if $0 \neq f \in Z - E$ and $\{f_j\}$ is a net in E such that $f_j \xrightarrow{w^*} f$ and $f_j \xrightarrow{s} p$, $p \in N_f \cap E$ then by the structural continuity of $|x|$, we get that $|f(x)| = |p(x)|$ for all x.

Therefore $f \in E$ and hence zero is the only w*-accumulation point of E. Hence X is a C_σ-space.

We now state three corollaries. We omit the easy proofs.

Corollary 4.2. *If X is a G-space then for any L-ideal N, $\bar{N}$ is an L-ideal. Moreover $(\bar{N})_1 = \overline{(N_1)}$ for all L-ideals N if and only if X is a C_σ-space.*

Corollary 4.3. *Let K be a compact convex set. The following are equivalent.*

(1) *K is a Bauer simplex.*

(2) (i) *$A = \{x \in E(K) : \{x\}$ is split$\}$ is dense in $\overline{E(K)}$,* (ii) *for any split face F, $\bar{F}$ is a split face.*

(3) *For all $a \in A(K)$, $|a| : E(K) \to \mathbf{R}$ is lower semi-continuous in the facial topology.*

REMARK. Corollary 4.3 improves a result of Å. Lima in [12].

COROLLARY 4.4. *A C_σ-space X is C_Σ iff E is compact in the structure topology.*

EXAMPLE 4.5. We now give an example of a separable G-space X and an L-ideal N in X such that for no $\lambda > 0$, $(\bar{N})_1 \subset \lambda\overline{(N_1)}$. This therefore furnishes a simple example of a subspace of characteristic zero in the sense of Dixmier [3].

Let $X = \{f \in C_{\mathbf{R}}[0,1]: f(1/n) = \bar{m}' f(1-1/n)$ for $n \geqq 3$ and $f(0) = 0 = f(1) = f(\frac{1}{2})\}$ then X is a G-space.

Let $Q:[0,1] \to X_1^*$ denote the evaluation map. Now it is not difficult to see that $E = \pm\{Q(x): x \in [0,1],\ x \neq 1/n,\ n \geqq 2\}$. Choose a sequence $\{x_n^i\}$ in $(0,\frac{1}{2})$ such that $\{x_n^i\}_{n=1}^\infty \cap \{1/i\}_{i=3}^\infty$ is empty for all $i \geqq 3$ and $x_n^i \to 1/i$ for all $i \geqq 3$.

Let $D = \bigcup_{i=3}^\infty \{Q(x_n^i)\}_{n=1}^\infty$. Then $D \subset E$ and $\bar{D} = D \cup \{1/i\}_{i=3}^\infty \cup \{0\}$. So if we let $N = \overline{\text{line}\, D}^{\text{Norm}}$, then N is an L-ideal and $\overline{(N_1)} = c(\pm D)$.

Let $B = \pm \bar{D} \cup \{Q(1-1/i)\}_{i=3}^\infty$. Since $Q(1-1/i) \xrightarrow{w^*} 0$ and $Q(1/i) = i^{-1}Q(1-1/i)$, $i \geqq 3$, we get that B is a dilated set. Therefore line $c(B)$ is a w*-closed L-ideal and $(\bar{N})_1 = (\text{line}\, c(B))_1 = c(B)$.

Suppose $\exists$ an integer $m_0 > 1$ and $c(B) \subseteq m_0 c(\pm D)$. Then $\{Q(1-1/i)\}_{i \geqq 3} \subset m_0 c(\pm D)$.

Fix $i \geqq 3$, get $f \in X$, $0 \leqq f \leqq 1$ such that $f(1-1/i) = 1$, $f(1/i) = 1/i$ and f peaks at $1/i$ in $(0,\frac{1}{2})$.

Now there exists a probability μ on $\bar{D} \cup -\bar{D}$ such that

$$\frac{1}{m_0} = \frac{Q(1-1/i)}{m_0}(f) = \int_{\bar{D} \cup -\bar{D}} f d\mu \leqq \frac{1}{i}$$

and this is true for any $i \geqq 3$. A contradiction.

Hence for no $\lambda > 1$, $(\bar{N})_1 \subset \lambda\overline{(N_1)}$.

THEOREM 4.6. *Let X be a complex Banach space such that any T-invariant set $D \subset E$ is structurally closed, then X is isometric to $c_0(\Gamma)$ (space of continuous functions vanishing at infinity on a discrete set Γ, with the supremum norm).*

PROOF. Use Theorem 4.3, to conclude that X is a C_σ-space. Let F be a maximal face of X_1^* and let $\Gamma = F \cap E$. Then $E = T\Gamma$ (see [15]).

Define $\Phi: X \to c_0(\Gamma)$ by

$$\Phi(x)(f) = f(x) \qquad \forall x \in X, \quad f \in \Gamma.$$

Then Φ is well defined (see proposition 4.8 of [5]), and an isometry.

To see that Φ is onto, let $\mathcal{T} \in c_0(\Gamma)$ and define $\mathcal{T}': E \cup \{0\} \to \mathbf{C}$ by $\mathcal{T}'(p) = t\mathcal{T}(q)$ if $p \in E$ and $p = tq$, $q \in F$, $t \in T$ and $\mathcal{T}'(0) = 0$. Now it is not hard to see that $\mathcal{T}'$ is well defined and is a w*-continuous, T-homogeneous function on $E \cup \{0\}$. Extend $\mathcal{T}'$, by Tietz's theorem, to a w*-continuous function g on X_1^* and let $h = \text{hom}\, g$.

Now by theorem 9 of [15], $\exists \nu \in X$ such that $f(\nu) = h(f)\ \forall f \in X_1^*$ and since h agrees with $\mathcal{T}'$ on E, we get that $f(\nu) = \mathcal{T}'(f)\ \forall f \in E \cup \{0\}$ and $\Phi(\nu) = \mathcal{T}$.

This completes the proof.

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